i. Let $z=2r$. $x+2y=6(n-r)$. y may have any value from 0 to $3(n-r)$.

$\therefore$ There are $3(n-r)+1$ solutions when z is $2r$.

$\therefore$ Total number of solutions for z even is

$$\sum_{r=0}^{r=n}[3(n-r)+1]=(3n+1)(n+1)-\frac{3n.(n+1)}{2}$$

$$=\frac{n+1}{2}(6n+2-3n)=\frac{(n+1)(3n+2)}{2}.$$

ii. Let $z=2r+1$. $x+2y=6(n-r)-3$. y may have any value from 0 to $3(n-r)-2$.

$\therefore 3(n-r)-2+1=3(n-r)-1$ solutions.

$\therefore$ Total number of solutions when z is odd:

$$=\sum_{r=0}^{r=n-1}[3(n-r)-1]=(3n-1)n-\frac{3n(n-1)}{2}=\frac{n}{2}(6n-2-3n+3)=\frac{n}{2}(3n+1).$$

$\therefore$ The total number of solutions

$$=\frac{(n+1)(3n+2)}{2}+\frac{n(3n+1)}{2}=\frac{3n^2+5n+2+3n^2+n}{2}=3n^2+3n+1.$$

Also solved by H. Prime, J. Scheffer, H. C. Feemster, and A. M. Harding.

363. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

(a) If a and n be positive integers, the integral part of $[a+\sqrt{(a^2-1)}]^n$ is odd.

(b) If a and n be positive integers, the integral part of $[\sqrt{(a^2+1)}+a]^n$ is odd when n is even and even when n is odd. [From Todhunter's *Algebra*, p. 353].

I. Solution by the PROPOSER.

Proof. (a) Let $[a+\sqrt{(a^2-1)}]^n=P+Q\sqrt{(a^2-1)}=m$.

Then $[a-\sqrt{(a^2-1)}]^n=P-Q\sqrt{(a^2-1)}=1/[a+\sqrt{(a^2-1)}]^n$.

$\therefore 0<P-Q\sqrt{(a^2-1)}<1$.

Adding m to each member of the inequality $m<2P<m+1$.

Therefore, the integral part of m is odd.

(b) Let $[\sqrt{(a^2+1)}+a]^n=R+S\sqrt{(a^2+1)}=k$.

Then $[-\sqrt{(a^2+1)}+a]^n=R-S\sqrt{(a^2+1)}=\left(\frac{-1}{\sqrt{(a^2+1)}+a}\right)^n$.

If n is even, $0<R-S\sqrt{(a^2+1)}<1$. Adding k, $k<2R<k+1$. Whence the integral part of k is odd.

If n is odd, $-1<R-S\sqrt{(a^2+1)}<0$. Adding k, $k-1<2R<k$. Whence the integral part of k is even.

Also solved similarly by S. Lefschetz.

II. Solution by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Burghfield, England, and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let I be the integral part of $[a+\sqrt{(a^2-1)}]^n$.
Then $I+F=[a+\sqrt{(a^2-1)}]^n$, where F is a proper fraction,

$$=a^n+na^{n-1}\sqrt{(a^2-1)}+\frac{n.(n-1)}{2!}(a^2-1)+\frac{n.(n-1)(n-2)}{3!}(a^2-1)^{\frac{3}{2}}+\ldots$$

Also, $a>\sqrt{(a^2-1)}$; $\therefore a-\sqrt{(a^2-1)}<1$, and $[a-\sqrt{(a^2-1)}]^n<1$.
Let $F'=[a-\sqrt{(a^2-1)}]^n=a^n-na^{n-1}\sqrt{(a^2-1)}$

$$+\frac{n.(n-1)}{2!}(a^2-1)-\frac{n.(n-1)(n-2)}{3!}(a^2-1)^{\frac{3}{2}}+\ldots$$

$$\therefore I+F+F'=2\left(a^n+\frac{n.(n-1)}{2!}(a^2-1)+\ldots\right)=2p \text{ (say)}.$$

Hence, $F+F'=1$, and $I=2p-1$ and is odd.

Similarly, $I+F=[\sqrt{(a^2+1)}+a]^n=(a^2+1)^{\frac{1}{2}n}+n(a^2+1)^{\frac{1}{2}(n-1)}$

$$+\frac{n.(n-1)}{2!}(a^2+1)^{\frac{1}{2}(n-1)}a^2+\ldots$$

$$F'=[\sqrt{(a^2+1)}-a]^n=(a^2+1)^{\frac{1}{2}n}-n(a^2+1)^{\frac{1}{2}(n-1)}a+\ldots$$

(i) n odd. $I+F-F'=2[n(a^2+1)^{\frac{1}{2}(n-1)}a+\ldots]=2p$ (say)=an even integer.

$\therefore F-F'=0$, and I is an even integer.

(ii) n even. $I+F+F'=2\left[(a^2+1)^{\frac{1}{2}n}+\frac{n.(n-1)}{2!}(a^2+1)^{\frac{1}{2}(n-2)}a^2+\ldots\right]$=an even integer.

$\therefore F+F'=1$, and I is an odd integer.